Lumiere SJPO Training
Advanced Handout I: Mathematics

0 Preface

"It's not that I'm so smart, it's just that I stay with problems longer"

Albert Einstein

This handout is the first in Lumiere's advanced series of notes. It assumes familiarity with **Secondary 4 O-Level Additional Mathematics**, and the notes are written with this background in mind.

However, please do not feel discouraged or be deterred if you are not yet Secondary 4 yet (or you've just started). You may still find it helpful and interesting to read through and ponder over this set of notes, and struggle through some of the problems. These topics are not as daunting as they seem; they get far more approachable with time, as you work through more problems using these concepts. That said, these ideas are foundational for more advanced physics anyway.

Take it slowly. Be patient with yourself. Struggle a little. And most of all, enjoy the journey.

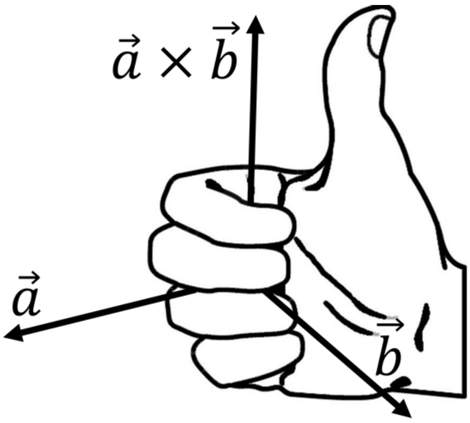
1 Introduction to Vectors

Vectors are quantities with both magnitude and direction. Some common vectors are velocity v, displacement s, and force F. Naturally, these quantities (sometimes) have their scalar counterparts, in speed u and distance d. Vectors are typically denoted by an arrow above the quantity ($\vec{X}$), a tilde below the quantity ($\underset{\sim}{E}$) or simply just bolded quantities ($\boldsymbol{B}$). In this handout, we will stick to using bolded quantities as vectors.

Vectors can also be added or multiplied to each other. We will focus on multiplication. The dot product is denoted by a dot ($\cdot$), and the cross product is denoted by a cross ($\times$). The dot product is also known as the scalar product of vectors, and the cross product is a vector product of vectors.

Dot products are formally defined as $\boldsymbol{a} \cdot \boldsymbol{b} = |a||b| \cos\theta$, where θ is the angle subtended by the two vectors. You can think of this as taking $\boldsymbol{a}$'s component parallel to $\boldsymbol{b}$ and multiplying those together. A common application of this is to find work done, where $W = \boldsymbol{F} \cdot \boldsymbol{d}$.

Cross products are defined as $\boldsymbol{a} \times \boldsymbol{b} = \hat{\boldsymbol{n}}\, |a||b| \sin\theta$. $\hat{\boldsymbol{n}}$ is a unit vector with magnitude 1, and its direction can be determined using your hands. There are numerous ways this is taught, but personally, I like to point my hand in the direction of $\boldsymbol{a}$ and curl my fingers in the direction of $\boldsymbol{b}$, where the direction of $\hat{\boldsymbol{n}}$ is given by the direction of your thumb.



2 Trigonometry

Some useful trigonometric identities are the following:

A. Addition formulae

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\tan(a+b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}$$

B. R-Formula

$$a\sin(\theta) + b\cos(\theta) = R\sin(\theta + \alpha)$$

$$R = \sqrt{a^2 + b^2}$$

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right)$$

C. Sum to product

$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

A neat way of remembering the sum to product formulae (although quite silly) is to replace the word sin with shop and cos with cash. Our mnemonic goes:

Shop + Shop = Shopping Centre
Shop – Shop = Close Shop
Cash + Cash = Credit Card
Cash – Cash = - (negative) Stop Shopping

For a triangle ABC with side lengths A, B, C and angles a, b, c, some geometrical properties you may find useful are:

A. Sine Rule

$$\frac{\sin(a)}{A} = \frac{\sin(b)}{B} = \frac{\sin(c)}{C}$$

B. Cosine Rule

$$c = \sqrt{A^2 + B^2 - 2AB\cos(c)}$$

C. Area

$$\text{Area} = \frac{1}{2}\text{AB}\sin(c)$$

3 Differentiation

3.1 The Basics

Suppose that we are given some function f(t) which describes the value of a quantity (let's say displacement) across time. You may then want to find another property, such as velocity which is the rate of change of the displacement. To do that you can simply take the derivative of f(t). We shall explore what this means more rigorously in the next sections and look at some techniques for differentiation. However, in essence, differentiating a function is simply finding the rate of change of the function or in other words how fast it is changing (with respect to a variable).

3.2 The definition of the derivative

Given a function f(x), we want to find the rate of change of f(x) at point x with respect to x. Graphically, we can draw a tangent line to the graph of y=f(x) at x and find its gradient. Alternatively, we can find the derivative of the function which would get you the gradient of the tangent line directly. Let's take advantage of a few properties. We should recognise that the tangent line is, well, tangent to the curve, meaning that around the point x $\pm\ \delta x$ (called delta x which is basically a small variation in x which can be as small as you want), the gradient of the tangent is equal to the gradient of the curve. This means that if you zoom in on the curve enough, such that δx is very small, the local region is ***approximately a straight line*** which gets closer and closer to being identical to a straight line as δx gets closer and closer to 0 (we call this act of getting closer and close to 0 the limit of δx as δx approaches 0 or in mathematical notation $\lim_{\delta x \to 0}(\delta x)$). Following this, we can find the gradient of the curve at that point just by using gradient = rise/run since we are essentially working with a straight line. Hence, gradient of f(x) at x = $\frac{d(f(x))}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = f'(x)$ (note that $\frac{dy}{dx}$ and $f'(x)$ are just different ways of writing the same thing).

The definition of the derivative:

$$\frac{d(f(x))}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

3.3 Finding some derivatives

Now that we have our shiny new formula, let's try to use it in some situations.

Example 3.3.1 – Finding the derivative of a quadratic function f(x)

Using the general expression for quadratics,

$$f(x) = a_2x^2 + a_1x + a_0$$

Substituting,

$$f'(x) = \lim_{\delta x \to 0} \frac{(a_2(x + \delta x)^2 + a_1(x + \delta x) + a_0) - (a_2x^2 + a_1x + a_0)}{\delta x}$$

Expanding the terms,

$$f'(x) = \lim_{\delta x \to 0} \frac{(a_2x^2 + 2a_2x\delta x + a_2\delta x^2 + a_1x + a_1\delta x + a_0 - a_2x^2 - a_1x - a_0)}{\delta x}$$

Eliminating terms,

$$f'(x) = \lim_{\delta x \to 0} 2a_2x + a_2\delta x + a_1$$

Evaluating the limit,

$$f'(x) = 2a_2x + a_1$$

Example 5.3.2 – Generalising our result to a degree-n polynomial

For simplicity we let's just look at the first term,

$$f(x) = a_nx^n$$

Substituting,

$$f'(x) = \lim_{\delta x \to 0} \frac{(a_n(x + \delta x)^n) - (a_nx^n)}{\delta x}$$

By the binomial theorem,

$$f'(x) = \lim_{\delta x \to 0} \frac{(\sum_{k=0}^{n}(a_n \binom{n}{k} x^{n-k}\delta x^k) - a_nx^n)}{\delta x}$$

Eliminating first term and dividing by δx,

$$f'(x) = \lim_{\delta x \to 0} (\sum_{k=1}^{n}(a_n \binom{n}{k} x^{n-k}\delta x^{k-1}))$$

Evaluating the limit (we can ignore all the other terms as they all go to 0),

$$f'(x) = (a_n \binom{n}{1} x^{n-1}\delta x^{1-1}))$$

Power rule: $\frac{d}{dx}(a_n x^n) = a_n n x^{n-1}$

This result is known as the power rule, one of the fundamental properties of differentiation

3.4 More rules and some essential derivatives

We shall simply summarise some of these rules to avoid this becoming a math class. You may refer to your O-Level Additional Mathematics Notes for further information.

Rules

Addition rule

$$\big(f(x) + g(x)\big)' = f'(x) + g'(x)$$

Product rule

$$\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x)$$

Quotient rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}$$

Chain rule

$$\big(f(g(x))\big)' = f'\big(g(x)\big) \cdot g'(x)$$

Essential derivatives

Polynomials

$$\frac{d}{dx}(a_n x^n) = a_n n x^{n-1}$$

Exponentials

$$\frac{d}{dx}(e^x) = e^x$$

Logarithms

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

Trigonometric functions

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$
$$\frac{d}{dx}(cos(x)) = -sin(x)$$
$$\frac{d}{dx}(tan(x)) = \sec^2(x)$$

3.5 Implicit differentiation

We will now quickly cover the concept of implicit differentiation which relies on the chain rule. Suppose that we are given an equation f(y) = g(x) where y itself is a function of x and we want to find the derivative of y with respect to x so $\frac{dy}{dx}$. As y is not explicitly expressed in terms of x we cannot simply differentiate the normal way, and we must use implicit differentiation.

We can first differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}f(y) = \frac{d}{dx}g(x)$$

Of course, we cannot just differentiate a function of y with respect to x. Hence, we must differentiate f(y) with respect to y first and multiply by the derivative of y with respect to x to maintain the equality on both sides of the equation.

$$\frac{d}{dy}f(y) \cdot \frac{dy}{dx} = \frac{d}{dx}g(x)$$

$$f'(y)\frac{dy}{dx} = g'(x)$$

Intuitively, this makes sense as $\frac{d(f(y))}{dx}$ should depend on both $f(y)$ itself and $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{g'(x)}{f'(y)}$$

Example 3.5.1

Given the equation $y^2 + (x-4)^2 = 16$, find $\frac{dy}{dx}$. (note that this is actually the equation of a circle)

Differentiating both sides with respect to x,

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(x-4)^2 = \frac{d}{dx}(16)$$

$$\frac{d}{dy}(y^2)\frac{dy}{dx} + 2(x-4) = 0$$

$$2y\frac{dy}{dx} = -2(x-4)$$

$$\frac{dy}{dx} = -\frac{(x-4)}{y}$$

3.6 Application of Differentiation: Force as the derivative of potential energy

In physics, we often deal with potentials since they are scalar quantities and hence easier to work with as they have no direction so you can just add them. From the potential, we may then use differentiation to find the force.

In general,

$$F(r) = -\frac{dU}{dr}$$

Where the force acts in the direction of decreasing potential as shown by the negative sign. If you are a little confused, this example below should clear things up.

Example 3.6.1 – Gravitational force (close to the ground)

We define upwards as the positive direction.

We all know from O-Level physics that the force is $-mg$ and the gravitational potential energy is mgh. Let's see if we can relate them using our new equation

$$U = mgh$$

$$F = -\frac{dU}{dh} = -mg$$

Example 3.6.2 – Gravitational force (very high up)

We define the direction pointing away from the centre of the earth as the positive direction.

From Newton's laws of gravitation (don't worry you will learn this soon),

$$U = -\frac{GMm}{r}$$

Where M is the mass of the Earth, r is the distance from the centre and m is the mass of the object

$$F = -\frac{dU}{dh}$$

$$F = -(-GMm(-1)r^{-2})$$

$$F = -\frac{GMm}{r^2}$$

3.7 Taylor approximations

In physics, we are sometimes given functions which are slightly annoying to deal with such as trigonometric functions (have fun trying to solve a differential equation with them). Therefore, it would be good if there was a way to replace them with a nicer function such as a polynomial. Luckily, the Taylor series exists to save our lives!! In essence, the Taylor series is about expressing a function as a polynomial where the n-th derivative of the series is equal to the n-th derivative of the actual function.

Let's now try to derive the full Taylor series for a function f(x) about x=0.

At x=0, the series has the same value as f(0) → the first term is $f(0)$

At x=0, the series has the same 1st derivative as f(x) → the 2nd term is $f'(0)x$

At x=0, the series has the same 2nd derivative as f(x) → the 3rd term is $\frac{f''(0)x^2}{2\times1}$

At x=0, the series has the same 3rd derivative as f(x) → the 4th term is $\frac{f'''(0)x^3}{3\times2\times1}$

At x=0, the series has the same 4th derivative as f(x) → the 5th term is $\frac{f'''(0)x^4}{4\times3\times2\times1}$

Hopefully you can see the pattern. If you're wondering why the denominators look like factorials (they are factorials) it is because of the repeated application of the power rule when taking higher order derivatives.

The general formula for a n-th order Taylor series about x=0 is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2\times1} + \frac{f'''(0)x^3}{3\times2\times1} + \cdots + \frac{f^n(0)x^n}{n!}$$

Equivalently,

$$f(x) = \sum_{k=0}^{n} \frac{f^k(0)}{k!}x^k$$

In physics, we usually only need to express it up to the **1st order term** as a common approximation that we work with is that x is small so higher order terms are negligible (try calculating 0.1 then 0.1^2 and 0.1^3 and so on). However there may be some cases where we need to look at up to the 3rd order and 99% of the time, we won't need anything higher than that.

Example 3.7.1

Find the Taylor approximation for sin(x) (about x=0) to the 3rd order.

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$
$$\frac{d^2}{dx^2}(\sin(x)) = -\sin(x)$$

$$\frac{d^3}{dx^3}(\sin(x)) = -\cos(x)$$

$$\sin(x) \approx \sin(0) + \frac{\cos(0)}{1!}x - \frac{\sin(0)}{2!}x^2 - \frac{\cos(x)}{3!}x^3$$

$$\sin(x) \approx x - \frac{x^3}{6}$$

4 Integration

4.1 The Basics

We have seen that differentiating a function with respect to its variable gives us its derivative. This raises the question of whether it is possible to recover the original function, given its derivative. Mathematically, integration can be viewed as such a process where the original function can be found given its derivative, albeit with some additional information required.

Suppose f and F are two functions related as follows:

$$\frac{dF(x)}{dx} = f(x)$$

Then we can write

$$dF(x) = f(x)\,dx$$ [1]

It is important to understand here that $dF(x)$ and dx represent a very small amount of $F(x)$ and x respectively. Taking the integral on both sides, we obtain

$$\int dF(x) = \int f(x)\,dx$$

If we are interested in the region where x changes from x_1 to x_2, we can introduce upper and lower limits on both sides of the integral, giving the definite integral

$$\int_{F(x_1)}^{F(x_2)} dF(x) = \int_{x1}^{x2} f(x)\,dx$$

A better way to present this would be

$$\int_{x_1}^{x_2} f(x)\,dx = [F(x)]_{x_1}^{x_2} = F(x_2) - F(x_1)$$ [2]

Some basic properties of definite integrals are as follows:

[1] A way to consider this is to "multiply the dx over". While considering d/dx as a fraction gives mathematically sound results and is commonly used in physics calculations, note that this lacks mathematical rigor and should NOT be used in areas outside of physics. d/dx should be considered as an operator instead.

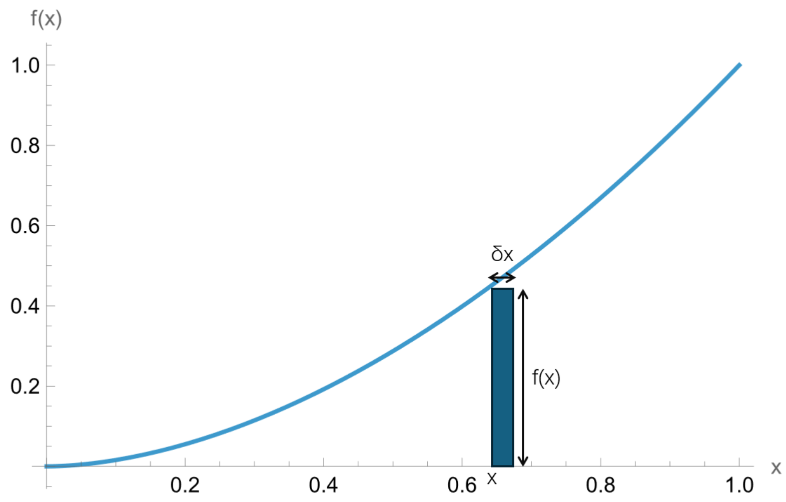
[2] For example, if $F(x) = x^2 + 1$ then $[x^2 + 1]_{-2}^{5} = 5^2 + 1 - [(-2)^2 + 1]$

$$\int_{x1}^{x2} f(x)dx = -\int_{x2}^{x1} f(x)\, dx$$

$$\int_{x1}^{x2} f(x)dx = \int_{x1}^{x3} f(x)\, dx + \int_{x3}^{x2} f(x)\, dx$$

(where x_3 is such that $x_1 \leq x_3 \leq x_2$)

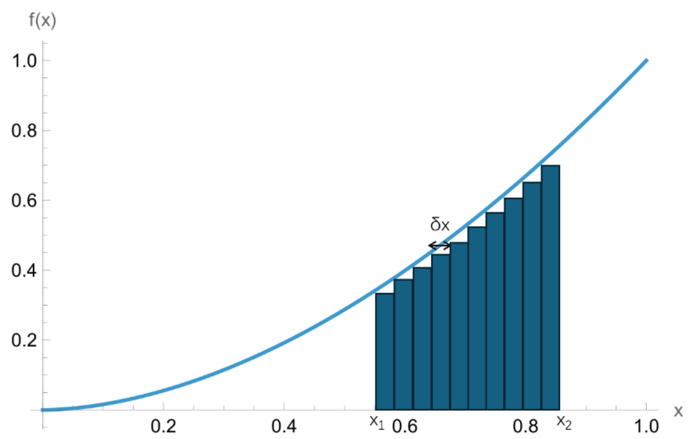
Another useful way to consider integration is finding the area under the graph of a function. To understand this, we use the following illustration:



For the graph of $f(x)$, we consider a "strip" at position x with thickness δx. The height of this strip is $f(x)$Thus, its area is $f(x)\, \delta x$.

For the region of interest x_1 to x_2, we can divide the graph into many such small strips of thickness δx. The sum of the areas of these strips can be approximated as

$$Area \approx \sum_{x1}^{x2} f(x)\, \delta\mathrm{x}$$



As δx decreases, the area of the strips becomes an increasingly better approximation for the actual area under the graph of $f(x)$ from $x_1$to x_2. Mathematically, we say that as δx approaches the limit of zero, the area under the graph becomes equal to the sum of the strips.

$$Area = \lim_{\delta x \to 0} \sum_{x1}^{x2} f(x)\, \delta x = \int_{x1}^{x2} f(x) dx$$

(The expression in the middle is mathematically defined to be equal to the definite integral of $f(x)$ from $x_1$to x_2)

This seemingly abstract definition can give integrals much more physical meaning. We will come back to this idea in the example questions later.

4.2 Solving integrals

Now that we are familiar with what an integral is, the next question is obviously: how do we solve these?

First, it is essential to recognise that no universal, systematic method exists for evaluating integrals. Unlike differentiation, which is governed by explicit and reliable rules such as the chain rule and product rule that can be applied even to highly complex expressions, integration lacks a comparable framework. As a result, many integrals resist standard techniques altogether, and oftentimes we cannot even find a closed form expression for an integral.

However, a comforting fact is that in most physics Olympiad questions, if a difficult integral is involved, the expression will be given in the question (after all it’s a physics Olympiad and not a Math Olympiad). Here, we will introduce some commonly used techniques that can make evaluating integrals less challenging.

Let us start with the forms of some common integrals (it is good to memorize these):

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c$$ [3]

$$\int sinx\, dx = -cosx + c$$

$$\int cosx\, dx = sinx + c$$

$$\int \frac{dx}{x} = \ln x + c$$

$$\int e^x\, dx = e^x + c$$

Integration by substitution

Another technique we will introduce is integration by substitution. How this works is we introduce a new variable to transform an integral involving the variable *x* to an integral involving a new variable of let's say, *u*.

Consider the integral $\int (5x+2)^3 dx$

We notice that this seems quite similar to the x^n form of integral above, but it's not quite the same. So, why not try letting $u = 5x + 2$?

Then,

$$\frac{du}{dx} = 5 \rightarrow \frac{dx}{du} = \frac{1}{5}$$

$$\therefore \int (5x+2)^3 dx = \int u^3 \frac{dx}{du} du$$

$$= \frac{1}{5} \int u^3 du$$

$$= \frac{1}{5}\left(\frac{u^4}{4}\right) + c$$

$$= \frac{(5x+2)^4}{20} + c$$

[3] Clearly, this is only valid for n≠-1. The c is an arbitrary constant called the constant of integration, and it is added to the end of indefinite integrals to represent the entire family of possible solutions (if you differentiate the right hand side, you can realize adding that adding a constant will not change the result).

We can see that by introducing a new variable, we can change an integral to a more familiar form. Here is a more challenging example:

Example 4.2.1

We aim to find $\int \frac{(\ln x)^3}{x} dx$.

It may not be immediately obvious what the substitution is here, but intuitively we can identify a few possibilities: $u = x$, $u = \ln x$, $u = (\ln x)^2$...

Clearly substituting $u = x$ is not the most helpful, so let's try the next option.

$$u = \ln x \rightarrow \frac{du}{dx} = \frac{1}{x} \rightarrow \frac{dx}{du} = x$$

$$\therefore \int \frac{(\ln x)^3}{x} dx = \int \frac{u^3}{x} \frac{dx}{du} du$$

$$= \int u^3 du$$

$$= \left(\frac{u^4}{4}\right) + c$$

$$= \frac{(\ln x)^4}{4} + c$$

4.3 Differential equations

Many of the fundamental laws of physics can be formulated as differential equations. These equations express relationships involving the rate of change of continuously changing quantities modelled by functions. For example, suppose a physicist observes that "the rate of decay of these particles is proportional to the number of particles at this time".

He may then formulate this mathematically. By letting the number of particles at any time be $N(t)$, he can write

$$\frac{dN(t)}{dt} = kN(t)$$

Where k is a negative constant.

This is a differential equation involving N as a function of t. If this physicist wants to predict the number of particles at any time, he will need to solve this equation to find $N(t)$.

Separation of variables

The main technique used in Physics to solve differential equations is called the separation of variables. This technique is GREATLY important.

As its name suggests, separation of variables involves manipulating the equation such that both sides of the equation can be independently integrated. A blunter way to put this would be to "lump all the related stuff on one side".

For example, for the abovementioned differential equation

$$\frac{dN}{dt} = kN$$

We can manipulate the equation by moving everything related to N on the left

$$\frac{dN}{N} = k\,dt$$

We can then integrate:

$$\int \frac{1}{N}\,dN = k\int dt$$

To remove the pesky constant of integration, we introduce boundary conditions to turn this into a definite integral. The boundary conditions are dictated by the physical context of the problem. In this example, the boundary condition can be "when $t = 0$, $N = N_0$". Then,

$$\int_{N_0}^{N} \frac{dN}{N} = k\int_0^t dt$$

Take note the matching of the upper and lower bounds on both sides of the equation. The upper bound represents an arbitrary time t and number of particles N. In words, we are "integrating time from 0 to t, where the number of particles changes from N_0 to N".

Solving:

$$[\ln N]\,\begin{matrix} N \\ N0 \end{matrix} = k[t]\,\begin{matrix} t \\ 0 \end{matrix}$$

$$\ln N - \ln N_0 = \ln\frac{N}{N_0} = kt$$

$$N = N_0 e^{kt}$$

This is the expression for $N(t)$ that we seek.

Now, let's look at some more examples that require a separation of variables.

Suppose a particle of mass m is driven by a force $F(t) = kt$. Given that the particle starts from rest, what is this particle's velocity as a function of time $v(t)$?

At first glance, some may naively try to apply the SUVAT equations to directly get $v(t)$. Using $v = u + at$ and $F = ma$, we can deduce that $a = \frac{kt}{m}$ and thus $v = \frac{kt^2}{m}$. However, this expression is incorrect! Take a quick pause here to consider why this is so.

And the answer is:

The SUVAT equations are *only valid when acceleration is constant*. Here, the acceleration is explicitly **time dependent**, which invalidates the SUVAT equations.

Here is the correct way to solve this problem:

From Newton's second law, we have

$$kt = ma = m\frac{dv}{dt}$$

We move everything related to t to one side:

$$t\,dt = \frac{m}{k}\,dv$$

$$\int_0^t t\,dt = \frac{m}{k}\int_0^v dv$$

$$\frac{t^2}{2} = \frac{m}{k}v$$

$$v = \frac{kt^2}{2m}$$

We can see that this equation is different from the answer we got by directly applying SUVAT by a factor of 2.

Let's up the difficulty further with the next example:

Suppose a particle of mass m is driven by a force $F(v) = kv$. Given that the particle starts from v_0, what is this particle's velocity as a function of time $v(t)$?

As with all kinematics problems, it is a good idea to start from $F = ma$.

$$kv = m\frac{dv}{dt}$$

This equation is very similar to the one we got from the decaying particles problem before. We go through the procedure of separating variables:

$$\frac{k}{m}\,dt = \frac{dv}{v}$$

$$\frac{k}{m}\int_0^t dt = \int_{v_0}^{v} \frac{dv}{v}$$

$$\ln\frac{v}{v_0} = \frac{k}{m}t$$

$$v = v_0 e^{\frac{kt}{m}}$$

Now we have derived the velocity-time relation for a time-dependent and velocity-dependent driving force, we can look at the last (and most difficult) example: a displacement-dependent driving force.

Suppose a particle of mass m is driven by a force $F(x) = kx$. Given that the particle starts from v_0 at $x = 0$, what is this particle's velocity *as a function of displacement* $v(x)$?

The question asking for a $v(x)$ relationship may feel intimidating, but we still start with the trusty

$$kx = m\frac{dv}{dt}$$

At this point, we realise we are stuck. The equation contains three variables—x, v, and t—and no obvious way to disentangle them. Perhaps the next step would be to eliminate t since we are looking for a $v(x)$ relation, but how?

After staring blankly at the equation as one might when first encountering this problem, we may feel motivated to write out a relationship between the three variables, i.e. $v = \frac{dx}{dt}$. With a small flash of inspiration, we then multiply both sides of the equation with dx.

$$kx\,dx = m\frac{dv}{dt}dx$$

$$kx\,dx = m\frac{dx}{dt}dv$$

$$x\,dx = \frac{m}{k}v\,dv$$

Wait, we have ended up with a differential equation with its variables separated!

$$\int_0^x dx = \frac{m}{k}\int_{v_0}^{v} v\,dv$$

$$x = \frac{m(v^2 - v_0^2)}{2k}$$

$$v = \sqrt{v_0^2 + \frac{2kx}{m}}$$

This example is a strong test of a student's physical intuition. Recognising the appropriate manipulation requires a deep understanding of how physical variables are related (but don't worry, physical intuition is built by doing more problems). While this trick itself is worth memorizing, the more important takeaway here is that when we are stuck, it is worth returning to fundamental definitions and relationships to search for motivation.

Practice 4.3.1:
A ball of mass m is dropped from a height h above the ground. As it falls, it experiences drag that is proportional to the velocity of the ball ($F_{drag} = kv$). How long does the ball take to reach the ground?

4.4 The concept of small quantities

In Physics, relationships between variables are seldom given to us in a neat form. More often, we must carefully analyse the physical system itself and formulate the governing differential equations from first principles. Central to this process is the ability to reason about small changes in quantities and how variables evolve locally, a skill that underpins the construction of differential equations and will absolutely be needed in the future.

A natural question to ask is: why focus on *small* quantities? How does this differ from analysing the system at a macroscopic level?

First, differential equations are inherently formulated in terms of infinitesimal changes. Quantities such as dx represent very small variations in x. Analysing the system locally therefore aligns directly with the mathematical language of differential equations and is often the most efficient approach. Also, when variables are small, powerful approximations become available. As discussed earlier in Taylor expansions, functions near zero can be linearised:

$$\sin x \approx \tan x \approx x$$

$$\cos x \approx 1$$

$$(1+x)^n \approx 1 + nx$$

These approximations can largely simplify the mathematical process of integration.

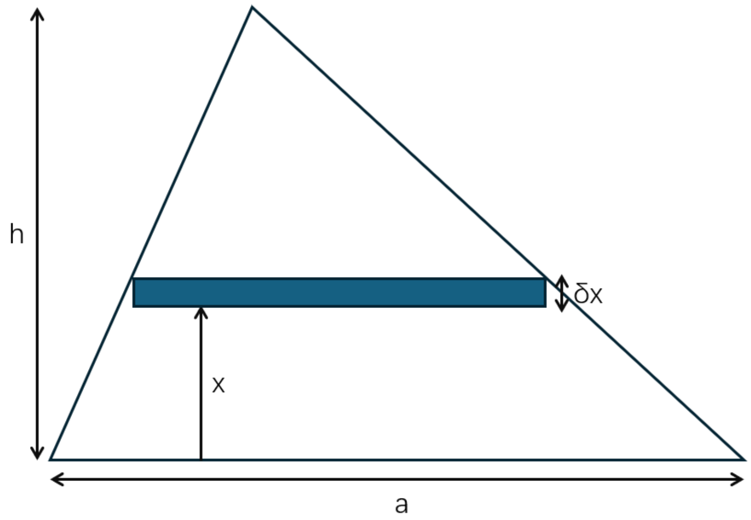
We have already had an example of small-quantity analysis when we introduced integration as calculating the area under the graph of a function. In that case, we split the region under the graph into many infinitesimally small strips and used a summation to find the area. Let's look at a few more examples:

Example 4.4.1

Prove that the area of a triangle $A = \frac{ah}{2}$ where a is the length of the base and h is the corresponding height.

There is a multitude of ways to prove this trivially, but we will intentionally obfuscate this problem by restricting ourselves to doing small quantity analysis.

We adopt a similar approach as before. We first cut a thin strip of thickness δx across the triangle that is a height x away from the base.



By similar triangles, the length of this strip is $\frac{a(h-x)}{h}$ and its area $\delta A = \frac{a(h-x)\delta x}{h}$. As $\delta x \to 0$, we can rewrite this as

$$dA = \frac{a(h-x)}{h}dx$$

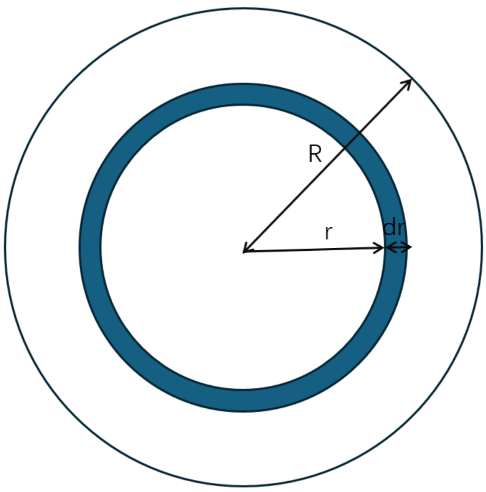
$$\int_0^A dA = \frac{a}{h}\int_0^h (h-x)dx$$

$$A = \frac{a}{h}\left[hx - \frac{x^2}{2}\right]_0^h = \frac{a}{h}\left(h^2 - \frac{h^2}{2}\right) = \frac{ah}{2}$$

Example 4.4.2

Prove that the area of a circle of radius R is $A = \pi R^2$.

Once again, we want to use the concept of small quantities.



One approach would be to cut a ring of radius r and thickness dr (I will skip over the part of taking $\delta r \to 0$ from now on) in the circle. Since dr is small, we can approximate the area of this ring to be $2\pi r\, dr$. Thus,

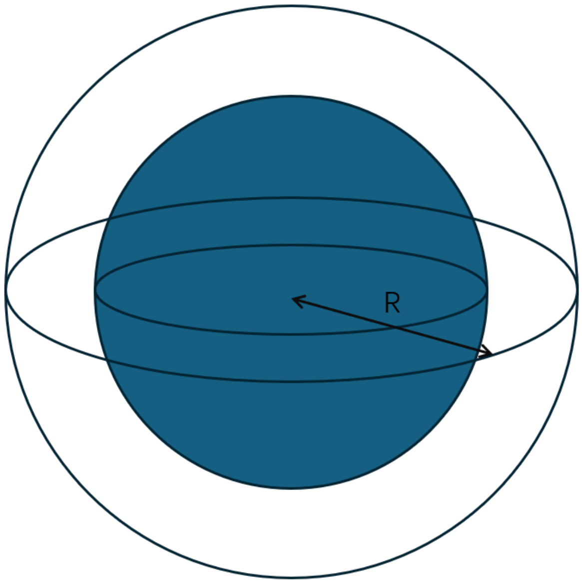
$$dA = 2\pi r\, dr$$
$$A = 2\pi \int_0^R r\, dr = \pi R^2$$

Right now, it seems that this concept has not much use except for making trivial problems harder. Then, let's look at an example which will be difficult to prove otherwise.

Example 4.4.3

Prove that the volume of a sphere of radius R is $V = \frac{4}{3}\pi R^3$.

(Hint: use surface area of sphere $= 4\pi R^2$)



This may take a bit of imagination, but the "strip" we are cutting this time will be a spherical shell of radius r and thickness dr (think of it like layers of an onion). The volume of this shell is

$$dV = 4\pi r^2\, dr$$
$$V = \int_0^R 4\pi r^2\, dr = \frac{4}{3}\pi R^3$$

Of course, there are also approaches that do not require the formula for the surface area of a sphere. The main purpose of the above three examples is to familiarize yourself with this type of analysis, which gets significantly more difficult when a physical system is considered.

Appendix A: Solving 2nd order linear ordinary differential equations

A common type of motion that we encounter in physics is oscillations. Many oscillating systems such as a spring mass system are governed by 2nd order linear ordinary differential equations. While it is unlikely that you will have to solve such equations in the SJPO, it is still useful to have this knowledge to give you a better understanding of oscillations (which we will cover later) and to make you a better physicist in general.

In a slightly unconventional approach, let's start and end with an example.

Example A – Simple oscillator

Suppose that we would like to investigate the motion of a block of mass m, connected to a spring with spring constant k.

Starting with Newton's 2nd Law,

$$m\ddot{x} = -kx$$

Note that $\ddot{x}$ is acceleration and $\dot{x}$ is velocity, (the number of dots basically means the order of the derivative so having 2 dots on x means that it is the 2nd order derivative of x which is acceleration)

Now, we should stare at the equation and think really hard of how we can find a function that satisfies this equation (basically a function, that when differentiated twice, can equal itself multiplied by some constant). You may remember that you have learnt of such a function before. It is indeed e^t ! (this is not a factorial, you're supposed to be happy that you discovered this, and if you're wondering why it's a t in the exponent, this is because it should be function of time)

Unfortunately (for both you and me since I now have to write more paragraphs), the answer is not as simple as this as we would get m = -k which is not really true. To repair our solution lets add some constants, making it $Ce^{\alpha t}$ where C and α are arbitrary constants.

We can try substituting this into our equation,

$$m\frac{d^2}{dx^2}(Ce^{\alpha t}) = -k(Ce^{\alpha t})$$

$$m\alpha^2(Ce^{\alpha t}) = -k(Ce^{\alpha t})$$

Cancelling and rearranging,

$$\alpha^2 = -\frac{k}{m}$$

At this point, you might panic and think that you made a mistake as it shouldn't be possible that α^2 is negative. But don't worry, this is a feature not a bug.

Let's just take a leap of faith and apply the square root anyway, factoring the -1 out,

$$\alpha = \pm\sqrt{-1}\sqrt{\frac{k}{m}}$$

We can call $\sqrt{-1}$ as i (the imaginary number – that's its actual name) so $\alpha = \pm i\sqrt{\frac{k}{m}}$

Hence, we have 2 solutions for x which are $x = C_1 e^{i\sqrt{\frac{k}{m}}t}$ and $x = C_2 e^{-i\sqrt{\frac{k}{m}}t}$

Making use of a property of such linear differential equations, we can actually add both the solutions to get the general solution[1].

So $x = C_1 e^{i\sqrt{\frac{k}{m}}t} + C_2 e^{-i\sqrt{\frac{k}{m}}t}$. This isn't very useful is it? Luckily Euler has a formula that will help us! ($e^{i\theta} = cos\theta + isin\theta$, yes this looks crazy but I promise its true)

Applying the formula, we get

$$x = C_1 cos\sqrt{\frac{k}{m}}\,t + iC_1 sin\sqrt{\frac{k}{m}}\,t + C_2 cos\sqrt{\frac{k}{m}}\,t - iC_2 sin\sqrt{\frac{k}{m}}\,t$$

$$x = (C_1 + C_2)cos\sqrt{\frac{k}{m}}\,t + i(C_1 - C_2)sin\sqrt{\frac{k}{m}}\,t$$

$$x = Acos(\omega t) + Bsin(\omega t)$$

Where $\omega = \sqrt{\frac{k}{m}}$

By looking at the expressions, we should notice that $\sqrt{\frac{k}{m}}$ will affect the frequency of the sine function, hence we call it the angular frequency ω (it is the Greek letter omega) where $\omega = 2\pi f$. A and B are arbitrary constants which are determined by the initial conditions of the system.

1 Let's say that x = f(t) and x = g(t) are 2 solutions of the equation. So $m \cdot f''(t) + kf(t) = 0$ and $m \cdot g''(t) + kg(t) = 0$, we can then add the 2 solutions together as follows, $m \cdot (f''(t) + g''(t)) + k(f(t) + g(t)) = 0$, to get the general solution which is $f(t) + g(t)$